

SPACES OF MATRICES OF CONSTANT RANK AND UNIFORM VECTOR BUNDLES.

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ABSTRACT. We consider the problem of determining $l(r, a)$, the maximal dimension of a subspace of $a \times a$ matrices of rank r . We first review, in the language of vector bundles, the known results. Then using known facts on uniform bundles we prove some new results and make a conjecture. Finally we determine $l(r; a)$ for every r , $1 \leq r \leq a$, when $a \leq 10$, showing that our conjecture holds true in this range.

INTRODUCTION.

Let A, B be k -vector spaces of dimensions a, b (k algebraically closed, of characteristic zero). A sub-vector space $M \subset \mathcal{L}(A, B)$ is said to be of (constant) rank r if every $f \in M, f \neq 0$, has rank r . The question considered in this paper is to determine $l(r, a, b) := \max \{\dim M \mid M \subset \mathcal{L}(A, B) \text{ has rank } r\}$. This problem has been studied some time ago by various authors ([21], [19], [4], [9]) and has been recently reconsidered, especially in its (skew) symmetric version ([16], [17], [15], [5]).

This paper is organized as follows. In the first section we recall some basic facts. It is known, at least since [19], that to give a subspace M of constant rank r , dimension $n + 1$, is equivalent to give an exact sequence: $0 \rightarrow F \rightarrow a.\mathcal{O}(-1) \xrightarrow{\psi} b.\mathcal{O} \rightarrow E \rightarrow 0$, on \mathbb{P}^n , where F, E are vector bundles of ranks $(a - r), (b - r)$. We observe that the bundle $\mathcal{E} := \text{Im}(\psi)$, of rank r , is *uniform*, of splitting type $(-1^c, 0^{r-c})$, where $c := c_1(E)$ (Lemma 2).

Then in Section two, we set $a = b$ to fix the ideas and we survey the known results (at least those we are aware of), giving a quick, uniform (!) treatment in the language of vector bundles. In Section three, using known results on uniform bundles, we obtain a new bound on $l(r; a)$ in the range $(2a + 2)/3 > r > (a + 2)/2$ (as well as some other results, see Theorem 18). By the way we don't expect this bound to be sharp. Indeed by "translating" (see Proposition 17) a long standing conjecture on uniform bundles (Conjecture 1), we conjecture that $l(r; a) = a - r + 1$ in this range (see Conjecture 2). Finally, with some ad hoc arguments, we show in the last section, that our conjecture holds true for $a \leq 10$ (actually we determine $l(r, a)$ for every r , $1 \leq r \leq a$, when $a \leq 10$).

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1. GENERALITIES.

Following [19], to give $M \subset \text{Hom}(A, B)$, a sub-space of constant rank r , with $\dim(M) = n + 1$, is equivalent to give on \mathbb{P}^n , an exact sequence:

$$(1) \quad 0 \longrightarrow F_M \longrightarrow a.\mathcal{O}(-1) \xrightarrow{\psi_M} b.\mathcal{O} \longrightarrow E_M \longrightarrow 0$$

where $\mathcal{E}_M = \text{Im}(\psi_M)$, F_M, E_M are vector bundles of ranks $r, a-r, b-r$ (in the sequel we will drop the index M if no confusion can arise).

Indeed the inclusion $i : M \hookrightarrow \text{Hom}(A, B)$ is an element of $\text{Hom}(M, A^\vee \otimes B) \simeq M^\vee \otimes A^\vee \otimes B$ and can be seen as a morphism $\psi : A \otimes \mathcal{O} \rightarrow B \otimes \mathcal{O}(1)$ on $\mathbb{P}(M)$ (here $\mathbb{P}(M)$ is the projective space of lines of M). At every point of $\mathbb{P}(M)$, ψ has rank r , so the image, the kernel and the cokernel of ψ are vector bundles.

A different (but equivalent) description goes as follows: we can define $\psi : A \otimes \mathcal{O}(-1) \rightarrow B \otimes \mathcal{O}$ on $\mathbb{P}(M)$, by $(v, \lambda f) \mapsto \lambda f(v)$.

The vector bundle \mathcal{E}_M is of a particular type.

Definition 1. A rank r vector bundle, E , on \mathbb{P}^n is uniform if there exists (a_1, \dots, a_r) such that $E_L \simeq \bigoplus_{i=1}^r \mathcal{O}_L(a_i)$, for every line $L \subset \mathbb{P}^n$ ((a_1, \dots, a_r) is the splitting type of E , it is independent of L).

The vector bundle F is homogeneous if $g^*(F) \simeq F$, for every automorphism of \mathbb{P}^n .

Clearly a homogeneous bundle is uniform (but the converse is not true).

The first remark is:

Lemma 2. With notations as in (1), $c_1(E_M) \geq 0$ and \mathcal{E}_M is a uniform bundle of splitting type $(-1^c, 0^b)$, where $c = c_1(E_M), b = r - c$.

Proof. Since E is globally generated, $c_1(E) \geq 0$ (look at E_L). Let $\mathcal{E}_L = \bigoplus \mathcal{O}_L(a_i)$. We have $a_i \geq -1$, because $a.\mathcal{O}_L(-1) \rightarrow \mathcal{E}_L$. We have $a_i \leq 0$, because $\mathcal{E}_L \hookrightarrow b.\mathcal{O}_L$. So $-1 \leq a_i \leq 0, \forall i$. Since $c_1(\mathcal{E}) = -c_1(E)$ the splitting type is as asserted and does not depend on the line L . \square

The classification of rank $r \leq n + 1$ uniform bundles on \mathbb{P}^n , $n \geq 2$, is known ([20], [10], [12], [1]):

Theorem 3. A rank $r \leq n + 1$ uniform vector bundle on \mathbb{P}^n , $n \geq 2$, is one of the following: $\bigoplus^r \mathcal{O}(a_i)$, $T(a) \oplus k.\mathcal{O}(b)$, $\Omega(a) \oplus k.\mathcal{O}(b)$ ($0 \leq k \leq 1$), $S^2 T_{\mathbb{P}^2}(a)$.

We will use the following result (see [8]):

Theorem 4. (Evans-Griffith)

Let \mathcal{F} be a rank r vector bundle on \mathbb{P}^n , then \mathcal{F} is a direct sum of line bundles if and only if $H_*^i(\mathcal{F}) = 0$, for $1 \leq i \leq r-1$.

The first part of the following Proposition is well known, the second maybe less.

Proposition 5. Assume $n \geq 1$.

- (1) If $a \geq b+n$ the generic morphism $a.\mathcal{O}_{\mathbb{P}^n} \rightarrow b.\mathcal{O}_{\mathbb{P}^n}(1)$ is surjective.
- (2) If $a < b+n$ no morphism $a.\mathcal{O}_{\mathbb{P}^n} \rightarrow b.\mathcal{O}_{\mathbb{P}^n}(1)$ can be surjective.

Proof. (1) It is enough to treat the case $a = b+n$ and, by semi-continuity, to produce one example of surjective morphism. Consider

$$\Psi = \begin{pmatrix} x_0 & \cdots & x_n & 0 & \cdots & 0 \\ 0 & x_0 & \cdots & x_n & 0 & \cdots & 0 \\ \vdots & \vdots & & & & & \\ 0 & \cdots & 0 & x_0 & \cdots & x_n \end{pmatrix}$$

(each row contains $b-1$ zeroes). It is clear that this matrix has rank b at any point. For a more conceptual (and complicated) proof see [14], Prop. 1.1.

(2) If $n = 1$, the statement is clear. Assume $n \geq 2$. If ψ is surjective we have $0 \rightarrow K \rightarrow a.\mathcal{O} \rightarrow b.\mathcal{O}(1) \rightarrow 0$ and K is a vector bundle of rank $r = a-b < n$. Clearly we have $H_*^i(K^\vee) = 0$ for $1 \leq i \leq r-1 \leq n-2$. By Evans-Griffith's theorem, K splits as a direct sum of line bundles, hence the exact sequence splits ($n \geq 2$) and this is absurd.

This can also be proved by a Chern class computation (see [19]). \square

From now on we will assume $A = B$ and write $l(r; a)$ instead of $l(r; a, a)$.

2. KNOWN RESULTS.

We begin with some general facts:

Lemma 6. Assume the bundle \mathcal{E} corresponding to $M \subset \text{End}(A)$ of constant rank r , $\dim(A) = a$, is a direct sum of line bundles. Then $\dim(M) \leq a-r+1$.

Proof. Let $\dim(M) = n+1$ and assume $\mathcal{E} = k.\mathcal{O}(-1) \oplus (r-k).\mathcal{O}$. If $k = 0$, the surjection $a.\mathcal{O}(-1) \rightarrow \mathcal{E} \simeq r.\mathcal{O}$, shows that $a \geq r+n$ (see Proposition 5). If $k > 0$, we have $0 \rightarrow k.\mathcal{O}(-1) \rightarrow (a-r+k).\mathcal{O} \rightarrow E \rightarrow 0$. Dualizing we get: $(a-r+k).\mathcal{O} \rightarrow k.\mathcal{O}(1)$, hence (always by Proposition 5) $a-r+k \geq k+n$. So in any case $a-r \geq n$. \square

Lemma 7. For every r , $1 \leq r \leq a$, we have $l(r; a) \geq a-r+1$.

Proof. Set $n = a - r$. On \mathbb{P}^n we have a surjective morphism $a.\mathcal{O}(-1) \xrightarrow{\bar{\psi}} r.\mathcal{O}$ (Proposition 5). Composing with the inclusion $r.\mathcal{O} \hookrightarrow r.\mathcal{O} \oplus (a - r).\mathcal{O}$, we get $\psi : a.\mathcal{O}(-1) \rightarrow a.\mathcal{O}$, of constant rank r . \square

Finally we get:

Proposition 8. (1) We have $l(r; a) \leq \max\{r + 1, a - r + 1\}$
(2) If $a \geq 2r$, then $l(r; a) = a - r + 1$.

Proof. (1) Assume $r + 1 \geq a - r + 1$. If $\dim(M) = l(r; a) = n + 1$ and if $r < n$, then ([12]) \mathcal{E} is a direct sum of line bundles and $n \leq a - r$. But then $r < n \leq a - r$, against our assumption. So $r + 1 \geq n + 1 = l(r; a)$.

If $a - r \geq r$. If $n > a - r$, then $n > r$ and this implies that \mathcal{E} is a direct sum of line bundles. Hence $n \leq a - r$.

(2) We have $\max\{r + 1, a - r + 1\} = a - r + 1$ if $a \geq 2r$. So $l(r; a) \leq a - r + 1$ by (1). We conclude with Lemma 7. \square

Remark 9. Proposition 8 was first proved (by a different method) by Beasley [4].

Very few indecomposable rank r vector bundles with $r < n$ are known on \mathbb{P}^n ($n > 4$). One of these is the bundle of Tango (see [18], p. 84 for details). We will use it to prove:

Lemma 10. We have $l(t + 1; 2t + 1) = t + 2$.

Proof. By Proposition 8 we know that $l(t + 1; 2t + 1) \leq t + 2$. So it is enough to give an example. Set $n = t + 1$ and assume first $n \geq 3$. If \mathcal{T} denotes the Tango bundle, then we have: $0 \rightarrow T(-2) \rightarrow (2n - 1).\mathcal{O} \rightarrow \mathcal{T} \rightarrow 0$. Dualizing we get $0 \rightarrow \mathcal{T}^\vee(-1) \rightarrow (2n - 1).\mathcal{O}(-1) \rightarrow \Omega(1) \rightarrow 0$. Combining with the exact sequence: $0 \rightarrow \Omega(1) \rightarrow (n + 1).\mathcal{O} \oplus (n - 2).\mathcal{O} \rightarrow \mathcal{O}(1) \oplus (n - 2).\mathcal{O} \rightarrow 0$, we get a morphism $(2n - 1).\mathcal{O}(-1) \rightarrow (2n - 1).\mathcal{O}$, of constant rank n .

If $n = 2$, using the fact that $T(-2) \simeq \Omega(1)$, from Euler's sequence, we get $3.\mathcal{O}(-1) \rightarrow 3.\mathcal{O}$, whose image is $T(-2)$. \square

Remark 11. Lemma 10 was first proved by Beasley ([4]), by a different method.

Finally on the opposite side, when r is big compared with a , we have:

Proposition 12. (Sylvester [19])

We have:

$$l(a - 1; a) = \begin{cases} 2 & \text{if } a \text{ is even} \\ 3 & \text{if } a \text{ is odd} \end{cases}$$

The proof is a Chern classes computation. The next case $a = r - 2$ is more involved and there are only partial results:

Proposition 13. (Westwick [23])

We have $3 \leq l(a - 2; a) \leq 5$. Moreover:

- (1) $l(a - 2; a, a) \leq 4$ except if $a \equiv 2, 10 \pmod{12}$ where it could be $l(a - 2; a, a) = 5$.
- (2) If $a \equiv 0 \pmod{3}$, then $l(a - 2; a, a) = 3$.
- (3) If $a \equiv 1 \pmod{3}$, we have $l(2; 4, 4) = 3$ and $l(8; 10, 10) = 4$ (so a doesn't determine $l(a - 2; a, a)$)
- (4) If $a \equiv 2 \pmod{3}$, then $l(a - 2; a, a) \geq 4$. Moreover if $a \not\equiv 2 \pmod{4}$, then $l(a - 2; a, a) = 4$.

Proof. We have $C(F) = C(E)(1 - h)^a$. Let $C(F) = 1 + s_1h + s_2h^2$, $C(E) = 1 + t_1h + t_2h^2$. We get $s_1 = t_1 - a$ (coefficient of h); $s_2 = a(a - 1)/2 - at_1 + t_2$ (coefficient of h^2). From the coefficient of h^3 it follows that: $t_2 = (a-1)[3t_1-a+2]/6$. The coefficient of h^4 yields after some computations: $(a+1)(a-2-2t_1) = 0$. It follows that $t_1 = \frac{(a-2)}{2}$ and $t_2 = \frac{(a-1)(a-2)}{12}$ if we are on $\mathbb{P}^n, n \geq 4$. Finally the coefficient of h^5 gives $(a+1)(a+2) = 0$, showing that $l(a - 2; a) \leq 5$.

If we are on \mathbb{P}^4 , from $t_1 = (a-2)/2$ we see that a is even. From $t_2 = (a-1)(a-2)/12$, we get $a^2 + 2 - 3a \equiv 0 \pmod{12}$. This implies $a \equiv 1, 2, 5, 10 \pmod{12}$. Since a is even we get $a \equiv 2, 10 \pmod{12}$.

If $a = 3m$ and if we are on \mathbb{P}^3 , then $6t_2 = (3m-1)(3t_1-3m+2) \equiv 0 \pmod{6}$, which is never satisfied. So $l(a - 2, a) \leq 3$ in this case.

The other statements follow from the construction of suitable examples, see [23].

□

Remark 14. On \mathbb{P}^4 a rank two vector bundle with $c_1 = 0$ has to verify the Schwarzenberger condition $c_2(c_2 + 1) \equiv 0 \pmod{12}$. If $l(a - 2; a) = 5$ for some a , then $a = 12m + 2$ or $a = 12m + 10$. In the first case the condition yields $m \equiv 0, 5, 8, 9 \pmod{12}$, in the second case $m \equiv 2, 3, 6, 11 \pmod{12}$. So, as already noticed in [23], the lowest possible value of a is $a = 34$. This would give an indecomposable rank two vector bundle with Chern classes $c_1 = 0, c_2 = 24$. Indeed if we have an exact sequence (1), E and F cannot be both a direct sum of line bundles (because, by Theorem 4, \mathcal{E} would also be a direct sum of line bundles, which is impossible).

Finally we have:

Proposition 15. (Westwick [22])

For every a, r , $l(r; a) \leq 2a - 2r + 1$

As noticed in [16] (Theorem 1.4) this follows directly from a result of Lazarsfeld on ample vector bundles. We will come back later on this bound.

3. FURTHER RESULTS AND A CONJECTURE.

There are examples, for every $n \geq 2$, of uniform but non homogeneous vector bundles on \mathbb{P}^n of rank $2n$ ([6]). However it is a long standing conjecture that every uniform vector bundle of rank $r < 2n$ is homogeneous. Homogeneous vector bundles of rank $r < 2n$ on \mathbb{P}^n are classified ([2]), so the conjecture can be formulated as follows:

Conjecture 1. *Every rank $r < 2n$ uniform vector bundle on \mathbb{P}^n is a direct sum of bundles chosen among: $S^2 T_{\mathbb{P}^2}(a)$, $\wedge^2 T_{\mathbb{P}^4}(b)$, $T_{\mathbb{P}^n}(c)$, $\Omega_{\mathbb{P}^n}(d)$, $\mathcal{O}_{\mathbb{P}^n}(e)$; where a, b, \dots, e are integers.*

The conjecture holds true if $n \leq 3$ ([10], [3]).

Before to go on we point out an obvious but useful remark.

Remark 16. *Clearly an exact sequence (1) exists if and only if the dual sequence twisted by $\mathcal{O}(-1)$ exists. So we may replace \mathcal{E} by $\mathcal{E}^\vee(-1)$. If \mathcal{E} has splitting type $(-1^c, 0^b)$, $\mathcal{E}^\vee(-1)$ has splitting type $(0^c, -1^b)$.*

Proposition 17. (1) *Take r, n such that $n \leq r < 2n$. Assume $a - r < n$ and that every rank r uniform bundle on \mathbb{P}^n is homogeneous. Then $l(r; a) \leq n$, except if $r = n, a = 2n - 1$ in which case $l(n; 2n - 1) = n + 1$.*

(2) *Assume Conjecture 1 is true. Then $l(r; a) = a - r + 1$ for $r < (2a + 2)/3$, except if $r = (a + 1)/2$, in which case $l(r; a) = a - r + 2$.*

Proof. (1) In order to prove the statement it is enough to show that there exists no subspace M of constant rank r and dimension $n + 1$ under the assumption $a - r < n$, $n \leq r < 2n$ (except if $r = n, a = 2n - 1$, in which case $l(n; 2n - 1) = n + 1$ by Lemma 10).

Such a space would give an exact sequence (1) with \mathcal{E} uniform of rank $r < 2n$ on \mathbb{P}^n . If \mathcal{E} is a direct sum of line bundles, by Lemma 6 we get $l(a; r) = a - r + 1 < n + 1$. Hence \mathcal{E} is not a direct sum of line bundles. Since the splitting type of \mathcal{E} is $(-1^c, 0^{r-c})$ (Lemma 2), we see that: $\mathcal{E} \simeq \Omega(1) \oplus k\mathcal{O} \oplus (r - k - n)\mathcal{O}(-1)$, $\mathcal{E} \simeq T(-2) \oplus t\mathcal{O} \oplus (r - t - n)\mathcal{O}(-1)$, or, if $n = 4$, $\mathcal{E} \simeq (\wedge^2 \Omega)(2)$.

Let's first get rid of this last case. The assumption $a - r < n$ implies $a \leq 9$. It is enough to show that there is no exact sequence (1) on \mathbb{P}^4 , with $\mathcal{E} = (\wedge^2 \Omega)(2)$ and $a = 9$. From $0 \rightarrow \mathcal{E} \rightarrow 9\mathcal{O} \rightarrow E \rightarrow 0$, we get $\mathcal{C}(E) = \mathcal{C}(\mathcal{E})^{-1}$. From the Koszul complex we have $0 \rightarrow \mathcal{E} \rightarrow \wedge^2 V \otimes \mathcal{O} \rightarrow \Omega(2) \rightarrow 0$. It follows that $\mathcal{C}(E) = \mathcal{C}(\Omega(2))$. Since $rk(E) = 3$ and $c_4(\Omega_{\mathbb{P}^4}(2)) = 1$, we get a contradiction.

So we may assume $\mathcal{E} \simeq \Omega(1) \oplus k.\mathcal{O} \oplus (r - k - n).\mathcal{O}(-1)$ or $\mathcal{E} \simeq T(-2) \oplus t.\mathcal{O} \oplus (r - t - n).\mathcal{O}(-1)$. By dualizing the exact sequence (1), we may assume $\mathcal{E} \simeq \Omega(1) \oplus k.\mathcal{O} \oplus (r - k - n).\mathcal{O}(-1)$. The exact sequence (1) yields:

$$0 \rightarrow \Omega(1) \oplus (r - n - k).\mathcal{O}(-1) \rightarrow (a - k).\mathcal{O} \rightarrow E \rightarrow 0 \quad (*)$$

Since $H_*^i(\Omega) = 0$ for $2 \leq i \leq n - 1$, from the exact sequence (*) we get $H_*^i(E) = 0$, for $1 \leq i \leq n - 2$. Since $rk(E) = a - r < n$, it follows from Evans-Griffith's theorem that $E \simeq \bigoplus \mathcal{O}(a_i)$. We have $a_i \geq 0, \forall i$, because E is globally generated. Moreover one a_i at least must be equal to 1 (otherwise $h^1(E^\vee \otimes \mathcal{E}) = 0$ and the sequence (*) splits, which is impossible). So $a_1 = 1, a_i \geq 0, i > 1$. It follows that $h^0(E) \geq (n + 1) + (a - r - 1) = n + a - r$. On the other hand $h^0(E) = a - k$ from (*).

If $k < r - n$, we see that one of the a_i 's, $i > 1$, must be > 0 . This implies $h^0(E) \geq 2(n + 1) + (a - r - 2) = 2n + a - r$. So $a - k = h^0(E) \geq 2n + a - r$. Since $a \geq a - k$, it follows that $a \geq 2n + a - r$ and so $r \geq 2n$, against our assumption.

We conclude that $k = r - n$ and $E = \mathcal{O}(1) \oplus (a - r - 1).\mathcal{O}$. In particular $\mathcal{E} = \Omega(1) \oplus (r - n).\mathcal{O}$ ((*) is Euler's sequence plus some isomorphisms). We turn now to the other exact sequence:

$$0 \rightarrow F \rightarrow a.\mathcal{O}(-1) \rightarrow \Omega(1) \oplus (r - n).\mathcal{O} \rightarrow 0 \quad (+)$$

We have $\mathcal{C}(F) = (1 - h)^a \cdot \mathcal{C}(\Omega(1))^{-1}$. Here $\mathcal{C}(F) = 1 + c_1 h + \dots + c_n h^n$ is the Chern polynomial of F (computations are made in $\mathbb{Z}[h]/(h^{n+1})$). From the Euler sequence $\mathcal{C}(\Omega(1))^{-1} = 1 + h$. It follows that:

$$\mathcal{C}(F) = (1 + h) \cdot \left(\sum_{i=0}^a \binom{a}{i} (-1)^i h^i \right)$$

Since $rk(F) = a - r < n$, $c_n(F) = 0$. Since $a \geq r \geq n$, it follows that $\binom{a}{n} = \binom{a}{n-1}$. This implies $a = 2n - 1$.

Observe that $r \geq n$ (because $k = r - n \geq 0$). If $r \geq n + 1$, then $rk(F) \leq n - 2$, hence $c_{n-1}(F) = 0$. This implies: $\binom{2n-1}{n-1} = \binom{2n-1}{n-2}$, which is impossible.

We conclude that $r = n$ and $a = 2n - 1$, so we are looking at $l(n; 2n - 1)$. By Lemma 10 we know that $l(n; 2n - 1) = n + 1$.

This proves (1).

(2) Now we apply (1) by setting $n := a - r + 1$. Clearly $n > a - r$. The condition $n \leq r < 2n$ translates in: $(a + 1)/2 \leq r < (2a + 2)/3$. So, under these assumptions, we get $l(r; a) \leq n = a - r + 1$, except if $r = n, a = 2n - 1$. In this latter case we know that $l(n; 2n - 1) = n + 1$ (Lemma 10). We conclude with Lemma 7. \square

Since Conjecture 1 is true for $r \leq n + 1$ and $n = 3, r = 5$ ([3]), we may summarize our results as follows:

Theorem 18.

- (1) If $r \leq a/2$, then $l(r, a) = a - r + 1$
- (2) If a is odd, $l(\frac{a+1}{2}; a) = \frac{a+1}{2} + 1 (= a - r + 2)$
- (3) If $\frac{(2a+2)}{3} > r \geq \frac{a}{2} + 1$, then $l(r; a) \leq r - 1$.
- (4) If a is even: $l(\frac{a}{2} + 1; a) = \frac{a}{2} (= a - r + 1)$.
- (5) If $r \geq (2a + 2)/3$, then $l(r, a) \leq 2(a - r) + 1$
- (6) We have $l(5; 7) = 3 (= a - r + 1)$.

Proof. (1) This is Proposition 8.

(2) This is Lemma 10.

(3) Set $n = r - 1$. Uniform vector bundles of rank $r = n + 1$ on \mathbb{P}^n are homogeneous. We have $n \leq r < 2n$ if $r \geq 3$ and $a - r < n$ if $r \geq (a/2) + 1$. If $r \leq 2$ and $r \geq (a/2) + 1$, then $a \leq 2$. Hence $r = a = 2$ and $l(2; 2) = 1$. So the assumption of Proposition 17, (1) are fulfilled. We conclude that $l(r, a) \leq r - 1$.

(4) Follows from (3) and Lemma 7.

(5) This is Proposition 15.

(6) Since uniform vector bundles of rank 5 on \mathbb{P}^3 are homogeneous, this follows from Proposition 17 (1) and Lemma 7. \square

Remark 19. Point 3 of the theorem improves the previous bound of Beasley but we don't expect this bound to be sharp (see Conjecture 2). Points 4 and 6 also are new. The bound of (5) is so far the best known bound in this range. It is reached for some values of a in the case $r = a - 1$ (Proposition 12), but already in the case $r = a - 2$ we don't know if it is sharp.

It is natural at this point to make the following:

Conjecture 2. Let a, r be integers such that $(2a + 2)/3 > r > (a/2) + 1$, then $l(r; a) = a - r + 1$.

Remark 20. This conjecture should be easier to prove than Conjecture 1, indeed in terms of vector bundles it translates as follows: every rank $r < 2n$ uniform vector bundle, \mathcal{E} , fitting in an exact sequence (1) on \mathbb{P}^n is homogeneous.

By the way the condition $r < 2n$ seems necessary. If $n = 2$ this can be seen as follows. Consider the following matrix (taken from [19]):

$$\Psi = \begin{pmatrix} 0 & -x_2 & 0 & -x_0 & 0 \\ x_2 & 0 & 0 & -x_1 & -x_0 \\ 0 & 0 & 0 & -x_2 & -x_1 \\ x_0 & x_1 & x_2 & 0 & 0 \\ 0 & x_0 & x_1 & 0 & 0 \end{pmatrix}$$

It is easy to see that Ψ has rank four at any point of \mathbb{P}^2 , hence we get:

$$0 \rightarrow \mathcal{O}(b) \rightarrow 5.\mathcal{O}(-1) \xrightarrow{\Psi} 5.\mathcal{O} \rightarrow \mathcal{O}(c) \rightarrow 0$$

with $\mathcal{E} = \text{Im}(\Psi)$ a rank four uniform bundle. On the line L of equation $x_2 = 0$, Ψ can be written:

$$\begin{aligned} \mathcal{O}_L(-3) \hookrightarrow & \quad 3.\mathcal{O}_L(-1) \twoheadrightarrow 2.\mathcal{O}_L \\ & \oplus \\ 2.\mathcal{O}_L(-1) \hookrightarrow & \quad 3.\mathcal{O}_L \twoheadrightarrow \mathcal{O}_L(2) \end{aligned}$$

It follows that $b = -3, c = 2$ and the splitting type of \mathcal{E} is $(-1^2, 0^2)$. Now rank four homogeneous bundles on \mathbb{P}^2 are classified (Prop. 3, p.18 of [7]) and are direct sum of bundles chosen among $\mathcal{O}(a), T(b), S^2T(c), S^3T(d)$. If \mathcal{E} is homogeneous the only possibility is $\mathcal{E}(1) \simeq T(-1) \oplus \mathcal{O}(1) \oplus \mathcal{O}$, but in this case the exact sequence $0 \rightarrow \mathcal{O}(-2) \rightarrow 5.\mathcal{O} \rightarrow \mathcal{E}(1) \rightarrow 0$, would split, which is absurd. We conclude that \mathcal{E} is not homogeneous. In fact \mathcal{E} is one of the bundles found by Elencwajg ([11]).

Remark 21. The results of this section and the previous one determine $l(r; a)$ for $a \leq 8, 1 \leq r \leq a$. To get a complete list for $a \leq 10$, we have to show, according to Conjecture 2, that $l(6; 9) = l(7; 10) = 4$. This will be done in the next section.

4. SOME PARTIAL RESULTS.

In the following lemma we relax the assumption $r < 2n$ in Proposition 17 when $c_1(\mathcal{E}(1)) = 1$.

Lemma 22. Assume we have an exact sequence (1) on \mathbb{P}^n , with $\text{rk}(F) = a - r < n$ and $c_1(\mathcal{E}(1)) = 1$. Then $a = 2n - 1$ and $r = n$.

Proof. If $c_1(\mathcal{E}(1)) = 1$, $\mathcal{E}(1)$ has splitting type $(1, 0^{r-1})$. It follows from [13], Prop. IV, 2.2, that $\mathcal{E}(1) = \mathcal{O}(1) \oplus (r-1).\mathcal{O}$ or $\mathcal{E}(1) = T(-1) \oplus (r-n).\mathcal{O}$. From the exact sequence $0 \rightarrow F(1) \rightarrow a.\mathcal{O} \rightarrow \mathcal{E}(1) \rightarrow 0$, we get $c_1(F(1)) = -1$. Since $F(1) \hookrightarrow a.\mathcal{O}$, it follows that $F(1)$ is uniform of splitting type $(-1, 0^{a-r-1})$. Since $\text{rk}(F) < n$, $F(1) = \mathcal{O}(-1) \oplus (a-r-1).\mathcal{O}$. This shows that necessarily $\mathcal{E}(1) = T(-1) \oplus (r-n).\mathcal{O}$. Now from the exact sequence: $0 \rightarrow T(-1) \oplus (r-n).\mathcal{O} \rightarrow a.\mathcal{O}(1) \rightarrow E(1) \rightarrow 0$, we get $\mathcal{C}(E(1)) = (\mathcal{C}(t(-1))^{-1}(1+h)^a$, i.e. $\mathcal{C}(E(1)) = (1-h)(1+h)^a$. Since $\text{rk}(E) < n$, we have $c_n(E(1)) = 0$ and arguing as in the proof of Proposition 17, we get $a = 2n - 1, r = n$. \square

Remark 23. Since we know that $l(n; 2n-1) = n+1$ (Lemma 10), we may, from now on, assume $c_1(\mathcal{E}(1)) \geq 2$.

Since $\mathcal{E}(1)$ is globally generated, taking $r-1$ general sections we get:

$$(2) \quad 0 \rightarrow (r-1).\mathcal{O} \rightarrow \mathcal{E}(1) \rightarrow \mathcal{I}_X(b) \rightarrow 0$$

Here X is a pure codimension two subscheme, which is smooth if $n \leq 5$ and which is irreducible, reduced, with singular locus of codimension ≥ 6 , if $n \geq 6$.

Lemma 24. *Assume $n \geq 3$ and $\text{rk}(F) < n$. If X is arithmetically Cohen-Macaulay (aCM), i.e. if $H_*^i(\mathcal{I}_X) = 0$ for $1 \leq i \leq n-2$, then F is a direct sum of line bundles.*

Proof. From (2) we get $H_*^i(\mathcal{E}) = 0$ for $1 \leq i \leq n-2$. By Serre duality $H_*^i(\mathcal{E}^\vee) = 0$, for $2 \leq i \leq n-1$. From the exact sequence $0 \rightarrow \mathcal{E}^\vee \rightarrow a.\mathcal{O}(1) \rightarrow F^\vee \rightarrow 0$, we get $H_*^i(F^\vee) = 0$, for $1 \leq i \leq n-2$. Since F^\vee has rank $< n$, by Evans-Griffith theorem we conclude that F^\vee (hence also F) is a direct sum of line bundles. \square

Proposition 25. *Assume that we have an exact sequence (1) on \mathbb{P}^4 with $\text{rk}(F) < 4$. Let $(-1^c, 0^{r-c})$ be the splitting type of \mathcal{E} . If $r > 4$ and if F is not a direct sum of line bundles, then $c, r-c \geq 4$; in particular $\text{rk}(\mathcal{E}) \geq 8$.*

Proof. Assume c or $b := r-c < 4$. By dualizing the exact sequence 1 if necessary, we may assume $b < 4$. We have an exact sequence (2):

$$0 \rightarrow (r-1).\mathcal{O} \rightarrow \mathcal{E}(1) \rightarrow \mathcal{I}_X(b) \rightarrow 0$$

where $X \subset \mathbb{P}^4$ is a smooth surface of degree $d = c_2(\mathcal{E}(1))$. If $b < 3$, X is either a complete intersection $(1, d)$ or lies on a hyper-quadric. In any case X is a.C.M. By Lemma 24, F is a direct sum of line bundles.

Assume $b = 3$. From the classification of smooth surfaces in \mathbb{P}^4 we know that if $d \leq 3$, then X is a.C.M. Now X is either a complete intersection $(3, 3)$, hence a.C.M. or linked to a smooth surface, S , of degree $9-d$ by such a complete intersection. If S is a.C.M. the same holds for X . From the classification of smooth surfaces of low degree in \mathbb{P}^4 , if X is not a.C.M. we have two possibilities:

- (i) X is a Veronese surface and S is an elliptic quintic scroll,
- (ii) X is an elliptic quintic scroll and S is a Veronese surface.

(i) If $X = V$ is a Veronese surface then we have an exact sequence:

$$0 \rightarrow 3.\mathcal{O} \rightarrow \Omega(2) \rightarrow \mathcal{I}_V(3) \rightarrow 0$$

It follows that $\mathcal{C}(\mathcal{E}(1)) = \mathcal{C}(\Omega(2)) = 1 + 3h + 4h^2 + 2h^3 + h^4$. So $\mathcal{C}(F(1)) = (\mathcal{C}(\mathcal{E}(1))^{-1} = 1 - 3h + 5h^2 - 5h^3$. It follows that F (and hence E also) has rank three. From $\mathcal{C}(E(1)) = (1+h)^a \cdot \mathcal{C}(F(1))$ and $c_4(E(1)) = 0$, we get $0 = a(a-5)(a-6)(a-7)$. So $a \leq 7$. Since $a = \text{rk}(E) + r$, we get a contradiction.

(ii) If $X = E$ is an elliptic quintic scroll, then we have:

$$0 \rightarrow T(-2) \rightarrow 5.\mathcal{O} \rightarrow \mathcal{I}_E(3) \rightarrow 0$$

It follows that $\mathcal{C}(\mathcal{E}(1)) = \mathcal{C}(T(-2))^{-1}$ and $\mathcal{C}(F(1)) = \mathcal{C}(T(-2)) = 1 - 3h + 4h^2 - 2h^3 + h^4$, in contradiction with $\text{rk}(F) < 4$. \square

Lemma 26. *Assume we have an exact sequence (1) on \mathbb{P}^4 with $a - r < 4$. If $r > 4$ and if F is a direct sum of line bundles, then $\text{rk}(\mathcal{E}) \geq 8$.*

Proof. If $r = 5$ we conclude with Theorem 18, (3), (6). If $r = 6$, then $a \leq 9$ and it is enough to show that $l(6; 9) \leq 4$ i.e. that there is no exact sequence (1) on \mathbb{P}^4 . In the same way, if $r = 7$ it is enough to show that $l(7; 10) \leq 4$.

If $r = 6$, we may assume that the splitting type of \mathcal{E} is $(-1^1, 0^5), (-1^2, 0^4), (-1^3, 0^3)$. By dualizing and by Lemma 22 we may disregard the first case. It follows that $c_1(F) = -7$ or -6 . If $r = 7$, in a similar way, we may assume that the splitting type of \mathcal{E} is $(-1^2, 0^5)$ or $(-1^3, 0^4)$. So $c_1(F) = -7$ or -8 .

Let $\mathcal{C}(F(1)) = (1 - f_1h)(1 - f_2h)(1 - f_3h)$. We have $\mathcal{C}(E(1)) = (1 + h)^a \mathcal{C}(F(1))$. From $c_4(E(1)) = 0$ we get:

$$\psi(a) := a^3 - a^2(4s + 6) + a(12d + 12s + 11) - 12d - 8s - 24t - 6 = 0$$

Where $s = -c_1(F(1)) = \sum f_i$, $d = c_2(F(1)) = \sum_{i < j} f_i f_j$, $t = -c_3(F(1)) = \prod f_i$. We have $f_i \geq 0, \forall i$ and $3 \leq s \leq 5$.

We have to check that this equality can't be satisfied for $a = 9, 10$. We have $\psi(9) = 8(42 - 28s + 12d - 37)$. If $\psi(9) = 0$ we get $3 \mid s$. It follows that $s = 3$. So the condition is: $4d - t = 14$. If one of the f_i 's is zero, then $t = 0$ and we get a contradiction. So $f_i > 0, \forall i$ and the only possibility is $(f_i) = (1, 1, 1)$, but then $4d - t = 11 \neq 14$.

For $a = 10$, we get $\psi(10) = 504 - 288s + 108d - 24t$. If $f_1 = f_2 = 0$, then $d = t = 0$ and we get $s = 504/288$ which is not an integer. If $f_1 = 0$, then $t = 0$, $d = f_2 f_3$, $s = f_2 + f_3$. If $s \geq 4$, $504 + 108d = 288s \geq 1152$. It follows that $d \geq 6$. If $d = 6$ we have necessarily $s = 5$ and $\psi(10) \neq 0$. So $s = 3$ and $d = 2$, but also in this case $\psi(10) \neq 0$. We conclude that $f_i > 0, \forall i$. So we are left with $(f_i) = (1, 1, 1), (1, 1, 2), (1, 1, 3), (1, 2, 2)$. In any of these cases one easily checks that $\psi(10) \neq 0$. \square

Corollary 27. *We have $l(6; 9) = l(7; 10) = 4$. In particular $l(r, a)$ is known for $a \leq 10$ and $1 \leq r \leq a$ and Conjecture 2 holds true for $a \leq 10$.*

Proof. We have seen that $l(6; 9), l(7; 10) \leq 4$, by Lemma 7 we have equality. Then all the other values of $l(r; a)$ are given by Theorem 18, Proposition 12 and Proposition 13, if $a \leq 10$. \square

REFERENCES

- [1] Ballico, E.: *Uniform vector bundles of rk $(n + 1)$ on \mathbb{P}^n* , Tsukuba J. Math., **7**, 215-226 (1983)
- [2] Ballico, E.-Ellia, Ph.: *Fibrés homogènes sur \mathbb{P}^n* , C. R. Acad. Sc. Paris, Série I, t.294, 403-406 (1982)
- [3] Ballico, E.-Ellia, Ph.: *Fibrés uniformes de rang 5 sur \mathbb{P}^3* , Bull. Soc. Math. France, **111**, n.1, 59-87 (1983)
- [4] Beasley, L.B.: *Spaces od matrices of equal rank*, Linear Algebra and its Applications, **38**, 227-237 (1981)

- [5] Boralevi, A.-Faenzi, D.-Mezzetti, E.: *Linear spaces of matrices of constant rank and instanton bundles*, Advances in Math., **248**, 895-920 (2013)
- [6] Drezet, J.M.: *Exemples de fibrés uniformes non homogènes sur \mathbb{P}^n* , C.R. Acad. Sc. Paris, Série A, t.**291**, 125-128 (1980)
- [7] Drezet, J.M.: *Fibrés uniformes de type $(0, \dots, 0, -1, \dots, -1)$ sur \mathbb{P}^2* , J. reine u. angew. Math., **325**, 1-27 (1981)
- [8] Ein, L.: *An analogue of Max Noether's theorem*, Duke Math. J., **52** (1985)
- [9] Eisenbud, D.-Harris, J.: *Vector spaces of matrices of low rank*, Advances in Math., **70**, 135-155 (1988)
- [10] Elencwajg, G.: *Les fibrés uniformes de rang 3 sur \mathbb{P}^2 sont homogènes*, Math. Ann., **231**, 217-227 (1978)
- [11] Elencwajg, G.: *Des fibrés uniformes non homogènes*, Math. Ann., **239**, 185-192 (1979)
- [12] Elencwajg, G.-Hirschowitz, A.-Schneider, M.: *Les fibrés uniformes de rang au plus n sur $\mathbb{P}^n(\mathbb{C})$ sont ceux qu'on croit*, in *vector bundles and differential equations*, Proc. Nice 1979, Progr. Math., **7**, 37-63, Birkhaeuser (1980)
- [13] Ellia, Ph.: *Sur les fibrés uniformes de rang $(n + 1)$ sur \mathbb{P}^n* , Mém. Soc. Math. France (Nouvelle série), **7**, (1982)
- [14] Ellia, Ph.-Hirschowitz, A.: *Voie ouest I: génération de certains fibrés sur les espaces projectifs et application*, J. Algebraic Geometry, **1**, 531-547 (1992)
- [15] Fania, M.L.-Mezzetti, E.: *Vector spaces of skew-symmetric matrices of constant rank*, Linear Algebra and its Applications, **434**, 2388-2403 (2011)
- [16] Ilic, B.-Landsberg J.M.: *On symmetric degeneracy loci, spaces of symmetric matrices of constant rank and dual varieties*, Math. Ann., **314**, 159-174 (1999)
- [17] Manivel, L.-Mezzetti, E.: *On linear space of skew-symmetric matrices of constant rank*, Manuscripta Math., **117**, 319-331 (2005)
- [18] Okonek, C.-Schneider, M.-Spindler, H.: *Vector bundles on complex projective spaces*, Progress in Math., **3**, Birkhaeuser (1980)
- [19] Sylvester, J.: *On the dimension of spaces of linear transformations satisfying rank conditions*, Linear Algebra and its App., **78**, 1-10 (1986)
- [20] Van de Ven, A.: *On uniform vector bundles*, Math. Ann., **195**, 245-248 (1972)
- [21] Westwick, R.: *Spaces of linear transformations of equal rank*, Linear Algebra and its Applications, **5**, 49-64 (1972)
- [22] Westwick, R.: *Spaces of matrices of fixed rank*, Linear and Multilinear Algebra, **20**, 171-174 (1987)
- [23] Westwick, R.: *Spaces of matrices of fixed rank, II*, Linear Algebra and its Applications, **235**, 163-169 (1996)

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